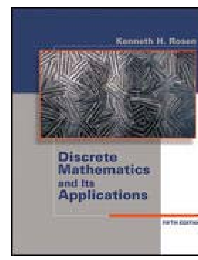
Chapter 2 (Part 4): The Fundamentals: Algorithms, the Integers & Matrices

- Matrices (Section 2.7)



- Introduction
  - Express relationship between elements in set
  - Solve large systems of equations
  - Useful in graph theory

– Definition 1

A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an m x n matrix. The plural of matrix is matrices. A matrix with the same number of rows as columns is called square. Two matrices are equal if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

– Example: The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ is a 3 X 2 matrix.

– Definition 2

Let $A = \begin{bmatrix} a_{11} a_{12} \ldots a_{1n} \\ a_{21} a_{22} \ldots a_{2n} \\ \vdots \quad \vdots \quad \vdots \\ a_{n1} a_{n2} \ldots a_{nn} \end{bmatrix}$.

The ith row of A is the 1 x n matrix $[a_{i1}, a_{i2}, \ldots, a_{in}]$. The jth column of A is the n x 1 matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

The (i, j)th element or entry of A is the element a_{ij}, that is, the number in the ith row and jth column of A. A convenient shorthand notation for expressing the matrix A is to write $A = [a_{ij}]$, which indicates that A is the matrix with its (i, j)th element equal to a_{ij}.

- Matrix Arithmetic

  – Definition 3

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be m x n matrices. The sum of A and B, denoted by A + B, is the m x n matrix that has $a_{ij} + b_{ij}$ as its (i, j)th element. In other words, $A + B = [a_{ij} + b_{ij}]$.

  – Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

  – Definition 4

Let A be an m x k matrix and B be a k x n matrix. The product of A and B, denoted by AB, is the m x n matrix with its (i, j)th entry equal to the sum of the products of the corresponding elements from the ith row of A and the jth column of B. In other words, if $AB = [c_{ij}]$, then

$$C_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}.$$

– Example: Let

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$$

Find AB if it is defined.

Solution: Since A is a 4 x 3 matrix and B is a 3 x 2 matrix, the product AB is defined and is a 4 x 2 matrix. To find the elements of AB, the corresponding elements of the rows of A and the columns of B are first multiplied and then these products are added. For instance, the element in the (3, 1)th position of AB is the sum of the products of the corresponding elements of the third row of A and the first column of B; namely 3 * 2 + 1 * 1 + 0 * 3 = 7. When all the elements of AB are computed, we see that

$$AB = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

– Example: Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Does AB = BA?

Solution: We find that

$$AB = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

Hence, AB ≠ BA.

- Matrix chain multiplication

  – Problem: How should the matrix-chain $A_1A_2\ldots A_n$ be computed using the fewest multiplication of integers, where $A_1A_2\ldots A_n$ are $m_1 \times m_2$, $m_2 \times m_3$, …, $m_n \times m_{n+1}$ matrices respectively and each has integers as entries?

– Example: $A_1 = 30 \times 20$ (30 rows and 20 columns)
$A_2 = 20 \times 40$
$A_3 = 40 \times 10$

Solution: ∃ 2 possibilities to compute $A_1A_2A_3$
1) $A_1(A_2A_3)$
2) $(A_1A_2)A_3$

1) First A_2A_3 requires 20 * 40 * 10 = 8000 multiplications
$A_1(A_2A_3)$ requires 30 * 20 * 10 = 6000 multiplications
Total: 14000 multiplications.

2) First A_1A_2 requires 30 * 20 * 40 = 24000 multiplications
$(A_1A_2)A_3$ requires 30 * 40 * 10 = 12000
Total: 36000 multiplications.

⇒ (1) is more efficient!

- Transposes and power matrices

  – Definition 5

The identity matrix of order n is the n x n matrix $I_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Hence

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

$$A^r = \underbrace{A * A * \dots * A}_{r\ times};\ A^0 = I_n$$

  – Definition 6

Let $A = [a_{ij}]$ be an m x n matrix. The transpose of A, denoted A^t, is the n x m matrix obtained by interchanging the rows and the columns of A. In other words, if $A^t = [b_{ij}]$, then $b_{ij} = a_{ij}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

  – Example:

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

– Definition 7

A square matrix A is called symmetric if $A = A^t$. Thus $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i and j with $1 \le i \le n$ and $1 \le j \le n$.

– Example: The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is symmetric.

- Zero-one matrices

  – It is a matrix with entries that are 0 or 1. They represent discrete structures using Boolean arithmetic.
  – We define the following Boolean operations:

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

– Definition 8

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be m x n zero-one matrices. Then the join of A and B is the zero-one matrix with (i, j)th entry $a_{ij} \vee b_{ij}$. The join of A and B is denoted $A \vee B$. The meet of A and B is the zero-one matrix with (i, j)th entry $a_{ij} \wedge b_{ij}$. The meet of A and B is denoted by $A \wedge B$.

– Example: Find the join and meet of the zero-one matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: We find that the joint of A and B is:

$$A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of A and B is:

$$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

– Definition 9

Let $A = [a_{ij}]$ be an m x k zero-one matrix and $B = [b_{ij}]$ be a k x n zero-one matrix. Then the Boolean product of A and B, denoted by $A \otimes B$, is the m x n matrix with (i, j)th entry $[c_{ij}]$ where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

– Example: Find the Boolean product of A and B, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution:

$$A \otimes B = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Algorithm The Boolean Product

```
procedure Boolean product (A,B: zero-one
  matrices)
for i := 1 to m
   for j := 1 to n
   begin
         c_ij := 0
         for q := 1 to k
              c_ij := c_ij ∨ (a_iq ∧ b_qj)
   end
{C = [c_ij] is the Boolean product of A and B}
```

– Definition 10

Let A be a square zero-one matrix and let r be a positive integer. The rth Boolean power of A is the Boolean product of r factors of A. The rth Boolean product of A is denoted by $A^{[r]}$. Hence

$$A^{[r]} = \underbrace{A \otimes A \otimes A \otimes \ldots \otimes A}_{r\ times}.$$

(This is well defined since the Boolean product of matrices is associative.) We also define $A^{[0]}$ to be I_n.

– Example: Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $A^{[n]}$ for all positive integers n.

Solution: We find that

$$A^{[2]} = A \otimes A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We also find that

$$A^{[3]} = A^{[2]} \otimes A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad A^{[4]} = A^{[3]} \otimes A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Additional computation shows that

$$A^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The reader can now see that $A^{[n]} = A^{[5]}$ for all positive integers n with $n \geq 5$.

- Exercises
  - 2a p.204
  - 4b p.204
  - 8 p.205
  - 28 p.206
  - 30 p.206