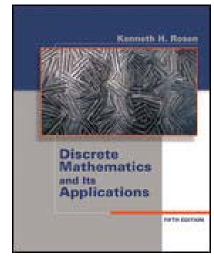
Ch.1 (Part 3): The Foundations: Logic and Proof, Sets, and Functions



- Set Operations (Section 1.7)
- Sequences, Summation, Cardinality of Infinites Sets (Section 1.7)

Set Operations (1.7) (cont.)

- Propositional calculus and set theory are both instances of an algebraic system called a

  Boolean Algebra.

  The operators in set theory are defined in terms of the corresponding operator in propositional calculus

  As always there must be a universe U. All sets are assumed to be subsets of U

Set Operations (1.7) (cont.)

- Definition:

Two sets A and B are equal, denoted A = B, iff $\forall x[x \in A \leftrightarrow x \in B]$.

- Note: By a previous logical equivalence we have

A = B iff $\forall x\, [(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$
or
A = B iff $A \subseteq B$ and $B \subseteq A$

Set Operations (1.7) (cont.)

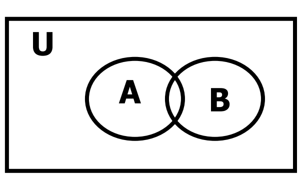
- Definitions:
  - The *union* of A and B, denoted A U B, is the set $\{x \mid x \in A \vee x \in B\}$
  - The *intersection* of A and B, denoted $A \cap B$, is the set
    $\{x \mid x \in A \wedge x \in B\}$
    Note: If the intersection is void, A and B are said to be *disjoint*.
  - The *complement* of A, denoted $\overline{A}$, is the set $\{x \mid \neg(x \in A)\}$
    Note: Alternative notation is A^c, and $\{x | x \notin A\}$.
  - The *difference* of A and B, or the *complement* of B *relative* to A, denoted A - B, is the set $A \cap \overline{B}$
    Note: The (absolute) complement of A is U - A.
  - The *symmetric difference* of A and B, denoted $A \oplus B$, is the set
    $(A - B) \cup (B - A)$

Set Operations (1.7) (cont.)

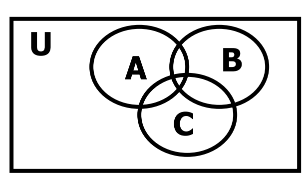
- **Examples:**
  U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
  A= {1, 2, 3, 4, 5},
  B = {4, 5, 6, 7, 8}. Then
  - $A \cup B$ = {1, 2, 3, 4, 5, 6, 7, 8}
  - $A \cap B$ = {4, 5}
  - $\overline{A}$ = {0, 6, 7, 8, 9, 10}
  - $\overline{B}$ = {0, 1, 2, 3, 9, 10}
  - A - B = {1, 2, 3}
  - B - A = {6, 7, 8}
  - $A \oplus B$ = {1, 2, 3, 6, 7, 8}

Set Operations (1.7) (cont.)

- Venn Diagrams
  - A useful geometric visualization tool (for 3 or less sets)
  - The Universe U is the rectangular box
  - Each set is represented by a circle and its interior
  - All possible combinations of the sets must be represented



For 2 sets



For 3 sets

  - Shade the appropriate region to represent the given set operation.

Set Operations (1.7) (cont.)

- Set Identities
  - Set identities correspond to the logical equivalences.
  - Example:

    The complement of the union is the intersection of the complements:

    $$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

    Proof: To show:

    $$\forall x\,[x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B}\,]$$

    To show two sets are equal we show for all x that x is a member of one set if and only if it is a member of the other.

Set Operations (1.7) (cont.)

- We now apply an important rule of inference (defined later) called

Universal Instantiation

In a proof we can eliminate the universal quantifier which binds a variable if we do not assume anything about the variable other than it is an arbitrary member of the Universe. We can then treat the resulting predicate as a proposition.

- We say

'Let x be arbitrary.'

Then we can treat the predicates as propositions:

Assertion	Reason
$x \in \overline{A \cup B} \Leftrightarrow x \notin [A \cup B]$	Def. of complement
$x \notin A \cup B \Leftrightarrow \neg[x \in A \cup B]$	Def. of $\notin$
$\Leftrightarrow \neg[x \in A \vee x \in B]$	Def. of union
$\Leftrightarrow \neg x \in A \wedge \neg x \in B$	DeMorgan's Laws
$\Leftrightarrow x \notin A \wedge x \notin B$	Def. of $\notin$
$\Leftrightarrow x \in \overline{A} \wedge x \in \overline{B}$	Def. of complement
$\Leftrightarrow x \in \overline{A} \cap \overline{B}$	Def. of intersection

Set Operations (1.7) (cont.)

Hence

$$x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B}$$

is a tautology.

Since

- x was arbitrary
- we have used only logically equivalent assertions and definitions

Set Operations (1.7) (cont.)

we can apply another rule of inference called

Universal Generalization

We can apply a universal quantifier to bind a variable if we have shown the predicate to be true for all values of the variable in the Universe.

and claim the assertion is true for all x, i.e.,

$$\forall x\ [x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B}\ ]$$

Q. E. D. (Latin phrase "Quod Erat Demonstrandum")

Set Operations (1.7) (cont.)

- Note: As an alternative which might be easier in some cases, use the identity

$$A = B \Leftrightarrow [A \subseteq B \text{ and } B \subseteq A]$$

- Example:

Show $A \cap (B - A) = \varnothing$

The void set is a subset of every set. Hence,

$$A \cap (B - A) \supseteq \varnothing$$

Therefore, it suffices to show

$A \cap (B - A) \subseteq \varnothing$ or $\forall x\ [x \in A \cap (B - A) \rightarrow x \in \varnothing]$

So as before we say 'let x be arbitrary'.

Set Operations (1.7) (cont.)

- Example (cont.)

  Show $x \in A \cap (B - A) \rightarrow x \in \varnothing$ is a tautology.

  But the consequent is always false.

  Therefore, the antecedent better always be false also.

  Apply the definitions:

Assertion	Reason
$x \in A \cap (B - A) \Leftrightarrow x \in A \wedge x \in (B - A)$	Def. of $\cap$
$\Leftrightarrow x \in A \wedge (x \in B \wedge x \notin A)$	Def. of $-$
$\Leftrightarrow (x \in A \wedge x \notin A) \wedge x \in B$	Props of 'and'
$\Leftrightarrow 0 \wedge x \in B$	Table 6
$\Leftrightarrow 0$	Domination

Set Operations (1.7) (cont.)

- Example (cont.)

  Hence, because $P \wedge \neg P$ is always false, the implication is a tautology.

  The result follows by Universal Generalization.

  Q. E. D.

Set Operations (1.7) (cont.)

- Union and Intersection of Indexed Collections
  - Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.
  - Union and intersection are associative (because 'and' and 'or' are) we have:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

and

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Set Operations (1.7) (cont.)

- **Examples**

Let $A_i = [i, \infty), 1 \leq i < \infty$

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$\bigcap_{i=1}^{n} A_i = [n, \infty)$

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$$\bigcap_{i=1}^{n} A_i = [n, \infty)$$

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- **Definition:** A ***sequence*** is a function from a subset of the natural numbers (usually of the form {0, 1, 2, . . . } to a set S.

  Note: the sets

  {0, 1, 2, 3, . . . , k} and {1, 2, 3, 4, . . . , k}

  are called *initial segments* of N.

  Notation: if f is a function from {0, 1, 2, . . .} to S we usually denote f(i) by a_i and we write

  $$\{a_0, a_1, a_2, ...\} = \{a_i\}_{i=0}^{k} = \{a_i\}_0^k$$

  where k is the upper limit (usually ∞).

Sequences, Summation, Cardinality of Infinites Sets (1.7)

Examples:

Using zero-origin indexing, if f(i) = 1/(i + 1). then the Sequence

f = {1, 1/'2,1/3,1/4, . . . } = {a_0, a_1, a_2, a_3, . . }

Using one-origin indexing the sequence f becomes

{1/2, 1/3, . . .} = {a_1, a_2, a_3, . . .}

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- Summation Notation

Given a sequence $\{a_i\}_0^k$ we can add together a subset of the sequence by using the summation and function notation

$$a_{g(m)} + a_{g(m+1)} + \ldots + a_{g(n)} = \sum_{j=m}^{n} a_{g(j)}$$

or more generally

$$\sum_{j \in S} a_j$$

Sequences, Summation, Cardinality of Infinites Sets (1.7)

Examples: $r^0 + r^1 + r^2 + \ldots + r^n = \sum_0^n r^j$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + = \sum_1^\infty \frac{1}{i}$$

$$a_{2m} + a_{2(m+1)} + \ldots + a_{2(n)} = \sum_{j=m}^{n} a_{2j}$$

$$\text{if } S = \{2,5,7,10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Similarity for the *product* notation: $\prod_{j=m}^{n} a_j = a_m a_{m+1} \ldots a_n$

Sequences, Summation, Cardinality of Infinites Sets (1.7)

Definition:

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \ldots .$$

Your book has a proof that

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1} \text{ if } r \neq 1$$

(you can figure out what it is if r = 1).

You should also be able to determine the sum

- if the index starts at k vs. 0
- if the index ends at something other than n (e.g., n-1, n+1, etc.).

Sequences, Summation, Cardinality of Infinites Sets (1.7) (cont.)

- Cardinality
  - **Definition:**

    The cardinality of a set A is equal to the cardinality of a set B, denoted

    $$|A| = |B|,$$

    if there exists a bijection from A to B.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- **Definition:**

  If a set has the same cardinality as a subset of the natural numbers N, then the set is called *countable*.

  If |A| = |N|, the set A is *countably infinite*.

  The (transfinite) cardinal number of the set N is *aleph null* = $\aleph_0$.

  If a set is not countable we say it is *uncountable*.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- **Examples:**

  The following sets are uncountable (we show later)

  - The real numbers in [0, 1]
  - P(N), the power set of N

- **Note:** With infinite sets proper subsets can have the same cardinality. This cannot happen with finite sets.

  Countability carries with it the implication that there is a *listing* of the elements of the set.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- **Definition:** | A | ≤ | B | if there is an injection from A to B.

  Note: as you would hope,

- **Theorem:**

  If | A | ≤ | B | and | B | ≤ | A | then | A | = | B |.

  This implies
  - if there is an injection from A to B
  - if there is an injection from B to A

    then
  - there must be a bijection from A to B

- This is <u>difficult</u> to prove but is an example of demonstrating existence without construction.
- It is often easier to build the injections and then conclude the bijection exists.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

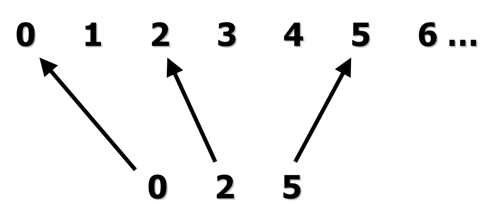
- **Example:**

  Theorem: If A is a subset of B then | A | ≤ | B |.

  Proof: the function f(x) = x is an injection from A to B.

- **Example:** {0, 2, 5}| ≤ $\aleph_0$

  The injection f: {0, 2, 5} → N defined by f(x) = x is shown below:



Sequences, Summation, Cardinality of Infinites Sets (1.7)

- Some Countably Infinite Sets
  - The set of even integers E (0 is considered even) is countably infinite. Note that E is a proper subset of N,

    Proof: Let f(x) = 2x. Then f is a bijection from N to E

0	**1**	**2**	**3**	**4**	**5**	**6 …**
↕	↕	↕	↕	↕	↕	↕
0	**2**	**4**	**6**	**8**	**10**	**12 …**

  - Z^+, the set of positive integers is countably infinite.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

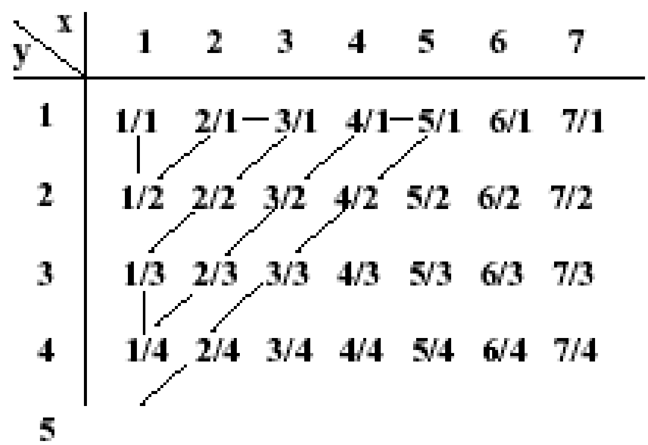
- The set of positive rational numbers Q^+ is countably infinite.

  Proof: Z^+ is a subset of Q^+ so $|Z^+| = \aleph_0 \le |Q^+|$.
  Now we have to show that $|Q^+| \le \aleph_0$.

  To do this we show that the positive rational numbers with repetitions, Q_R, is countably infinite.

  Then, since Q^+ is a subset of Q_R, it follows that $|Q^+| \le \aleph_0$ and hence $|Q^+| = \aleph_0$.

Sequences, Summation, Cardinality of Infinites Sets (1.7)



Sequences, Summation, Cardinality of Infinites Sets (1.7)

- The position on the path (listing) indicates the image of the bijective function f from N to Q_R:

  f(0) = 1/1, f(1) = 1/2, f(2) = 2/1, f(3) = 3/1, and so forth.

  Every rational number appears on the list at least once, some many times (repetitions).

  Hence, $|N| = |Q_R| = \aleph_0$. Q. E. D

- The set of all rational numbers Q, positive and negative, is countably infinite.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- The set of (finite length) strings S over a finite alphabet A is countably infinite.

  To show this we assume that
  - A is nonvoid
  - There is an "alphabetical" ordering of the symbols in A

  Proof: List the strings in lexicographic order:
  - all the strings of zero length,
  - then all the strings of length 1 in alphabetical order,
  - then all the strings of length 2 in alphabetical order,

  etc.

  This implies a bijection from N to the list of strings and hence it is a countably infinite set.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- For **example**:

  Let A = {a, b, c}.

  Then the lexicographic ordering of A is

  {λ , a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab, aac, aba, ...} = {f(0), f(1), f(2), f(3), f(4), ...}

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- The set of all C programs is *countable*.

  Proof: Let S be the set of legitimate characters which can appear in a C program.

  - A C compiler will determine if an input program is a syntactically correct C program (the program doesn't have to do anything useful).
  - Use the lexicographic ordering of S and feed the strings into the compiler.
    - If the compiler says YES, this is a syntactically correct C program, we add the program to the list.
    - Else we move on to the next string.
- In this way we construct a list or an implied bijection from N to the set of C programs.
- Hence, the set of C programs is countable. Q. E. D.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- Cantor Diagonalization
  - An important technique used to construct an object which is not a member of a countable set of objects with (possibly) infinite descriptions

    Theorem: The set of real numbers between 0 and 1 is uncountable.
    Proof: We assume that it is countable and derive a contradiction.

    If it is countable we can list them (i.e., there is a bijection from a subset of N to the set).

    We show that no matter what list you produce we can construct a real number between 0 and 1 which is not in the list.

    Hence, there cannot exist a list and therefore the set is not countable

Sequences, Summation, Cardinality of Infinites Sets (1.7)

It's actually much bigger than countable. It is said to have the *cardinality of the continuum*, c.

Represent each real number in the list using *its decimal expansion.*

e.g., 1/3 = .3333333........
1/2 = .5000000........
= .4999999........

If there is more than one expansion for a number, it doesn't matter as long as our construction takes this into account.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- THE LIST....

$r_1 = .d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}. \; . \; . \; . \; .$
$r_2 = .d_{21}d_{22}d_{23}d_{24}d_{25}d_{26} \; . \; . \; . \; .$
$r_3 = .d_{31}d_{32}d_{33}d_{34}d_{35}d_{36} \; . \; . \; . \; .$
...

Now construct the number $x = .x_1x_2x_3x_4x_5x_6x_7. \; . \; . \; .$

$x_i = 3 \text{ if } d_{ii} \neq 3$
$x_i = 4 \text{ if } d_{ii} = 3$

(Note: choosing 0 and 9 is not a good idea because of the non uniqueness of decimal expansions.)

Then x is not equal to any number in the list.

Hence, no such list can exist and hence the interval (0,1) is uncountable. Q. E. D.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

- An extra goody:

  Definition: a number x between 0 and 1 is *computable* if there is a C program which when given the input i, will produce the ith digit in the decimal expansion of x.

- **Example:**

  The number 1/3 is computable.

  The C program which always outputs the digit 3, regardless if the input, computes the number.

Sequences, Summation, Cardinality of Infinites Sets (1.7)

Theorem: There is exists a number x between 0 and 1 which is *not computable*.

There *does not exist* a C program (or a program in any other language) which will compute it!

Why? Because there are more numbers between 0 and 1 than there are C programs to compute them.

(in fact there are c such numbers!)

Our second example of the *nonexistence* of programs to compute things!